Analytical Model for Tidal Waves Dynamics Predictions in the Convergent Estuaries Involving Bottom Friction

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Abstract

The analytical solution of the one-dimensional non-linear partial differential equation for tidal wave propagation in the estuary can be determined using the perturbation method. The solution is based on the St Venantequation which consists of the continuity and momentum equation. This research was conducted at the mouth of the KarangMumus-Mahakam river which is assumed to have no bottom slope. A cross-differentiation between conservation of mass and conservation of momentum equations results in the hyperbolic differential equation. The solution is only limited to the second-order and the solution can also be reviewed through the completion of graphic visualization where the solution provides a decrease in deviation from the estuary to the end of the estuary. The deviation of the water level from the average water level gives the potential for brackish water to be utilized to develop aquaculture activities.

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1. Introduction

The estuary is semi-enclosed coastal water, which has access to the open sea and contains seawater that is sufficiently measurable, its area extends to river areas, is affected by tides, and in this area, seawater is mixed with freshwater from rivers on land significantly[1].

Efforts to find an analytical solution of the viscosity flow equation were started by Berre' de Saint Venant in 1843. Analytical solutions are one of the options for converting nonlinear partial differential equations to be easy to apply. In a flow system, the general equation can be simplified more easily than in a tidal water bodies. Rivers tend to have a more or less constant cross-section. Another important discovery was solving the St. Venants equation for estuaries with exponentially varying cross-sections. In this study, an analytical solution was developed using a one-dimensional model, considering several assumptions on topography and flow characteristics. In tidal hydraulics, the cross-section is usually assumed to be constant (rectangular, trapezoidal, or triangular) or variable. This general assumption is also used by Savenije[2].

In 1998 Savenije used the Lagrangian approximation and obtained a nonlinear amplification equation by subtracting the highest water layer (HW) and the lowest water layer (LW) which maintains the quadratic velocity and periodic variation of the hydraulic radius [3]. This method is hereinafter referred to as the envelope method, which is a quasi-nonlinear approach because it uses a linear harmonic function which is a linearization method. Based on this quasi-nonlinear approach, an explicit solution of the tidal hydraulic equation is found, by solving a series of four implicit equations [4].

An analytical approach is also applied to other studies to predict freshwater discharge in the estuary based on observations of tidal levels [5]. With the same model, an explicit solution is also presented and applied to the mouth of the SebouMarocco river [6]. This model is then applied to the estuary of the KarangMumus river, East Kalimantan by assuming the estuary area is generated by tides with a channel that has a bottom slope and does not consider lateral flow then analytically uses the method of solving non-linear partial differential equations (that is by using the perturbation method). order zero [7]. In this research, the perturbation method is applied to solve non-linear partial differential equations (nonlinear St. Venant equations) in the convergent estuaries of the KarangMumus Delta Mahakam river which is affected by the presence of bottom friction with a very small base slope and the solution approximates up to the third-order.

2. Basic Equation

Conceptual sketches of the present model of tidal wave propagation generated by tidal force in an convergent estuary is presented in **Figure 1**, x is an axis of propagation as longitudinal coordinates measured from the river mouth, h is the depthof an estuary, z is level of water fluctuation, h_0 is the mean water depth at the river mouth, HW is level of high water and LW is level of low water. A sketch of the waves in the estuary area is after [8]as follow.

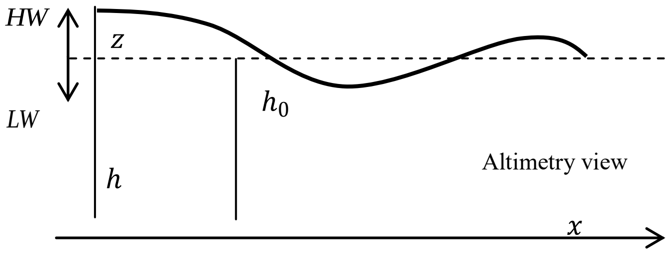

Figure 1. Sketches of waves in anconvergent estuaries.

The equation used in this study is a one-dimensional Saint Venant equation, assuming that long waves are generated by tides, flowing into a constant channel width with the varied depth to be smaller than the width, no freshwater discharge, compared to tidal discharge, and Froude number is set to be about two. Some parameters are used as follow: a tidal period T is set to be the semi-diurnal tide period,a velocity V, η is a fluctuating deviation from mean water level, ρ as the water densityis set to be homogenous, g is the acceleration due to gravity, C_h is Chezy's friction factor as external force from the bottom, A is the cross-sectional area of the considered channel, and$Q = AV$ as the tidal flow discharge.

The governing equation consists of conservation of mass and momentum equations. Thesetwo equations are the main equations of the SaintVenantfor open channels to explain the tidal dynamics in the estuary in one-dimensional equations as follows

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0, \tag{1}$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial z_b}{\partial x} + g\frac{\partial h}{\partial x} - g\frac{h}{2\rho}\frac{\partial \rho}{\partial x} + g\frac{V|V|}{C_h^{2}h} = 0. \tag{2}$$

3. Scaling the equations

In this section, a dimensionless processis an important physical processto guarantee that to develop a model will has no a dimension problem during a solving process afterward. If a model is assumed to have a length variable x then the dimensionless length variable is x^*can be stated following [9], where $[x]$ is relevant length

$$x = [x]x^*. \tag{3}$$

In this section dimensionless parameters can be identified which can be used to describe the characteristics of tidal waves that occur in the river estuary channel by taking certain constant values, where the corresponding reference value of the main quantity is adopted as a constant scale. The dimensionless process is stated in **Table 1**as follows.

Table 1 Non-dimensional processes.

Dimensional variables	Dimensionless variables
$x = h_0 x^*$	x^*
$t = t^* / \omega$	t^*
$V = V^* \sqrt{gh_0}$	V^*
$h = h_0\, h^*$	h^*
$h = h_0 + \eta$	$h^* = 1 + \eta^*$

This dimensionless form is needed to compare the terms in the equation when looking for a solution using a perturbation method. With this dimensionless form it can also be known which componenthas more significant valuein line with solving process, where each subsequent solution will maintain to be closer to the exact solution than the solution previously obtained.
The dimensionless process of equations (1) and (2) give the following results

$$\omega \frac{\partial h^*}{\partial t^*} + \frac{\sqrt{g}}{\sqrt{h_0}} \frac{\partial (h^* V^*)}{\partial x^*} = 0, \tag{4}$$

$$\omega \frac{\partial V^*}{\partial t^*} + \frac{\sqrt{g}}{\sqrt{h_0}} V^* \frac{\partial V^*}{\partial x^*} + \frac{\sqrt{g}}{\sqrt{h_0}} \frac{\partial h^*}{\partial x^*} + \frac{\sqrt{g}}{\sqrt{h_0}} \left(S - S_\rho + S_f \right) = 0. \tag{5}$$

4. The perturbation method

The perturbation theory in mathematics can be used to get a solution approach to complex differential equations and when the exact solution is hard to find. In general, small disturbances in the physical system are denoted by ε, where ε is a perturbation parameter that has very little value or can be lesser than one. The way to get a solution approach to the perturbation method is to use matched asymptotic expansion. The asymptotic expansion method is defined as follow

Definition-1

Function $\phi_1(\varepsilon), \phi_2(\varepsilon), \cdots$forms an asymptotic row for $\varepsilon \to \varepsilon_0$if and if only $\phi_{m+1} = O(\phi_m)$ for $\varepsilon \to \varepsilon_0$for every m

Definition-2

If $\phi_1(\varepsilon), \phi_2(\varepsilon), \cdots$ is asymptotic row, then $f(\varepsilon)$ istheasymptotic expansion for n-th term, if and if only

$f = \sum_{k=1}^{m} a_k \phi_k + O(\phi_m)$, for $m = 1, 2, \cdots, n$; and $\varepsilon \to \varepsilon_0$, Coefficient a_k doesnot depend on ε , consequently it can be written as

$$f \approx a_1\phi_1(\varepsilon)+\cdots+a_n\phi_n(\varepsilon) \text{ for } \varepsilon \to \varepsilon_0 . \tag{6}$$

Functions ϕ_k are said as scale or gauge function [10].Gauge function is positive, monotone function and stands on an interval where $0<\varepsilon<\varepsilon_0$. The simplest form of gauge function and the most used is the power of ε , namely $1,\varepsilon,\varepsilon^2,\varepsilon^3,\cdots$.

5. Solution of long wave equation

The conservation of mass conservation equation and momentum conservation equations in equations (3) and (4) can be uniformly written as

$$a\frac{\partial h^*}{\partial t^*}+\frac{\partial\left(h^*V^*\right)}{\partial x^*}=0, \tag{7}$$

$$a\frac{\partial V^*}{\partial t^*}+V^*\frac{\partial V^*}{\partial x^*}+\frac{\partial h^*}{\partial x^*}+S_f=0. \tag{8}$$

The valueof $a=\omega\sqrt{h_0/g}$ is constant.Both equations (7) and (8) will be modified to obtain a long wave equation. The first step is to multiply V^* on the equation (7) and multiply h^* on the equation obtained (8)

$$aV^*\frac{\partial h^*}{\partial t^*}+V^*\frac{\partial(h^*V^*)}{\partial x^*}=0, \tag{9}$$

$$ah^*\frac{\partial V^*}{\partial t^*}+h^*V^*\frac{\partial V^*}{\partial x^*}+h^*\frac{\partial h^*}{\partial x^*}+h^*S_f=0. \tag{10}$$

Add these two equations above, yield

$$a\left(\frac{\partial}{\partial t^*}\left(h^*V^*\right)\right)+\frac{\partial}{\partial x^*}\left(h^*V^{*2}\right)+h^*\frac{\partial h^*}{\partial x^*}+h^*S_f=0. \tag{11}$$

Differentiating equation (7) with respect to t^* and equation (11) with respect to x^*, so it can be resulted a new equation

$$a^2\frac{\partial^2 h^*}{\partial t^{*2}}+a\frac{\partial^2\left(h^*V^*\right)}{\partial x^*\partial t^*}=0, \tag{12}$$

$$a\frac{\partial^2\left(h^*V^*\right)}{\partial x^*\partial t^*}+\frac{\partial^2}{\partial x^{*2}}\left(h^*V^{*2}\right)+\frac{\partial h^*}{\partial x^*}\frac{\partial h^*}{\partial x^*}+h^*\frac{\partial^2 h^*}{\partial x^{*2}}+S_f\frac{\partial h^*}{\partial x^*}=0. \tag{13}$$

By cross-differentiation of equations (7) and (11), equations (12) and (13) yield

$$a^2\frac{\partial^2 h^*}{\partial t^{*2}}=\frac{\partial^2}{\partial x^{*2}}\left(h^*V^{*2}\right)+\frac{\partial h^*}{\partial x^*}\frac{\partial h^*}{\partial x^*}+h^*\frac{\partial^2 h^*}{\partial x^{*2}}+S_f\frac{\partial h^*}{\partial x^*}. \tag{14}$$

Because $h^* = (1+\eta^*)$ and $V^* = (1+\eta^*)$, so the equation (14) can be written as

$$a^2 \frac{\partial^2 \eta^*}{\partial t^{*2}} = 3\frac{\partial^2 \eta^*}{\partial x^{*2}} + 3\eta^* \frac{\partial^2 \eta^*}{\partial x^{*2}} + 3\left(\frac{\partial \eta^*}{\partial x^*}\right)^2 + S_f \frac{\partial \eta^*}{\partial x^*} \qquad (15)$$

The next step is to use an assumption related to the form of asymptotic expansion. Here, the asymptotic expansion is written in the form of

$$\eta^*(x^*,t^*) = \varepsilon^1 \eta_0^* + \varepsilon^2 \eta_1^* + \varepsilon^3 \eta_2^* \qquad (16)$$

atau

$$\eta^*(x^*,t^*) = \varepsilon^1\left(\varepsilon^0 \eta_0^* + \varepsilon^1 \eta_1^* + \varepsilon^2 \eta_2^*\right) \qquad (17)$$

where $\varepsilon^0, \varepsilon^1, \varepsilon^2$ is a series of gauge functions, $\eta^*(x^*,t^*)$ is a function of variable x^* and t^* which represents solutions for zeroth, first-order, and-th-order solutions. Substitute equation (17) into equation (15):

$$a^2 \frac{\partial^2\left(\varepsilon^0 \eta_0^* + \varepsilon^1 \eta_1^* + \varepsilon^2 \eta_2^*\right)}{\partial t^{*2}} = 3\frac{\partial^2\left(\varepsilon^0 \eta_0^* + \varepsilon^1 \eta_1^* + \varepsilon^2 \eta_2^*\right)}{\partial x^{*2}} + S_f \frac{\partial\left(\varepsilon^0 \eta_0^* + \varepsilon^1 \eta_1^* + \varepsilon^2 \eta_2^*\right)}{\partial x^*} +$$
$$3\varepsilon\left(\varepsilon^0 \eta_0^* + \varepsilon^1 \eta_1^* + \varepsilon^2 \eta_2^*\right)\frac{\partial^2\left(\varepsilon^0 \eta_0^* + \varepsilon^1 \eta_1^* + \varepsilon^2 \eta_2^*\right)}{\partial x^{*2}} + 3\varepsilon\left(\frac{\partial\left(\varepsilon^0 \eta_0^* + \varepsilon^1 \eta_1^* + \varepsilon^2 \eta_2^*\right)}{\partial x^*}\right)^2 \qquad (18)$$

Equation (18) can be written in the form of a zero-order linear equation $O(1)$, first-order $O(\varepsilon)$, second-orde $O(\varepsilon^2)$:

$$O(1): a^2 \frac{\partial^2 \eta_0^*}{\partial t^{*2}} - 3\frac{\partial^2 \eta_0^*}{\partial x^{*2}} - S_f \frac{\partial \eta_0^*}{\partial x^*} = 0 \qquad (19)$$

$$O(\varepsilon^1): a^2 \frac{\partial^2 \eta_1^*}{\partial t^{*2}} - 3\frac{\partial^2 \eta_1^*}{\partial x^{*2}} - S_f \frac{\partial \eta_1^*}{\partial x^*} = 3\eta_0^* \frac{\partial^2 \eta_0^*}{\partial x^{*2}} + 3\left(\frac{\partial \eta_0^*}{\partial x^*}\right)^2 \qquad (20)$$

$$O(\varepsilon^2): a^2 \frac{\partial^2 \eta_2^*}{\partial t^{*2}} - 3\frac{\partial^2 \eta_2^*}{\partial x^{*2}} - S_f \frac{\partial \eta_2^*}{\partial x^*} = 3\eta_0^* \frac{\partial^2 \eta_1^*}{\partial x^{*2}} + 3\eta_1^* \frac{\partial^2 \eta_0^*}{\partial x^{*2}} + 6\frac{\partial \eta_0^*}{\partial x^*}\frac{\partial \eta_1^*}{\partial x^*} \qquad (21)$$

on condition that the boundary is at the mouth of the estuary $\eta^*(0) = 0.155$ and $\eta^*(\infty) = 0$. Each of the above equations can be separated by variables so that the solution $\eta^*(x^*,t^*) = \eta^*(x^*)\eta^*(t^*)$, Therefore, we obtain a dimensional solution of the form:

$$\eta(x,t) = \eta(x)\eta(t) \qquad (22)$$

Zero-order equation $O(1)$ (18) can be written:

$$a^2 \eta^*_{0t^*t^*} - 3\eta^*_{0x^*x^*} - S_f \eta^*_{0x^*} = 0 \qquad (23)$$

If the solution η_0^* that depends on t is $\approx e^{i\omega t}$, then the solution η_0^* that depends on t^* can be written as follows:

$\approx e^{it^*}$ (24)

By substituting the second derivative of equation (23) into equation (22), we get:

$$3\eta^*_{0x^*x^*} + S_f \eta^*_{0x^*} + a^2 \eta^*_0 = 0 \quad (25)$$

Example solution $\eta_0^*(x^*) = e^{rx^*}$, obtained the characteristic equation:

$$3r^2 + S_f r + a^2 = 0 \quad (26)$$

Assuming that $\sqrt{S_f^{\,2} - 12a^2}$ the real value is positive, so the general solution of equation (25) is

$$\eta_0^* = C_1 e^{\frac{-S_f + \sqrt{S_f^{\,2} - 12a^2}}{6}x} + C_2 e^{\frac{-S_f - \sqrt{S_f^{\,2} - 12a^2}}{6}x} \quad (27)$$

and

$$f = \frac{-S_f + \sqrt{S_f^{\,2} - 12a^2}}{6} \text{ and } j = \frac{-S_f - \sqrt{S_f^{\,2} - 12a^2}}{6}$$

and obtained the boundary conditions at the mouth of the estuary for zero-order are $\eta_0^*(0) = 0.155$ and $\eta_0^*(1000) = 0$. By entering the boundary conditions, we get

$$\eta_0^*(0) = C_1 e^0 + C_2 e^{0x} = 0.155 \quad (28)$$

$$\eta_0^*(1000) = C_1 e^{f(1000)} + C_2 e^{j(1000)} = 0 \quad (29)$$

Substituting equation $C_1 = 0.155 - C_2$ into equation (29) we get

$$C_2 = \frac{-0.155 e^{f(1000)}}{e^{j(1000)} - e^{f(1000)}} \quad (30)$$

To get C_1 substitution C_2 to C_1

$$C_1 = 0.155 + \frac{0.155 e^{f(1000)}}{e^{j(1000)} - e^{f(1000)}} \quad (31)$$

The special zero-order solution $O(1)$ of equation (23) is

$$\eta_0^* = \left(0.155 + \frac{0.155 e^{f(1000)}}{e^{j(1000)} - e^{f(1000)}}\right) e^{fx^*} + \frac{-0.155 e^{f(1000)}}{e^{j(1000)} - e^{f(1000)}} e^{jx^*} \quad (32)$$

The derivatives η_0^* can be written as follows:

$$\eta_0^* = C_1 e^{fx^*} + C_2 e^{jx^*}$$

$$\frac{\partial \eta_0^*}{\partial x^*} = f C_1 e^{fx^*} + j C_2 e^{jx^*} \quad (33)$$

$$\frac{\partial^2 \eta_0^*}{\partial x^{*2}} = f^2 C_1 e^{fx^*} + j^2 C_2 e^{jx^*}$$

Determination of the first-order solution η_1^* depends on the form of the zero-order solution η_0^* by substituting equations (32) and (33) into equations (19) and $S = 0$, the result is:

$$-3\frac{\partial^2\eta_1^*}{\partial x^{*2}} - S_f\frac{\partial\eta_1^*}{\partial x^*} - a^2\eta_1^* = 6f^2C_1^{\ 2}e^{2fx^*} + \left(3f^2C_1C_2 + 6fC_1jC_2 + 3j^2C_2C_1\right)e^{fx^*+jx^*} + 6j^2C_2^{\ 2}e^{2jx^*} \quad (34)$$

The homogeneous equation of equation (34) is $3\frac{\partial^2\eta_1^*}{\partial x^{*2}} + S_f\frac{\partial\eta_1^*}{\partial x^*} + a^2\eta_1^* = 0$, in the same way as in the previous step obtained homogeneous solution

$$\eta_{1h}^* = C_3e^{\frac{-S_f+\sqrt{S_f^{\ 2}-12a^2}}{6}x} + C_4e^{\frac{-S_f-\sqrt{S_f^{\ 2}-12a^2}}{6}x} \quad (35)$$

Based on equation (34) choose a nonhomogeneous solution

$$\eta_{1p}^* = Ae^{fx^*+jx^*} + Be^{2fx^*} + Ce^{2jx^*} \quad (36)$$

Substituting equation (36) and its derivatives into equation (34), we get

$$\eta_{1p}^* = Ae^{fx^*+jx^*} + Be^{2fx^*} + Ce^{2jx^*}$$
$$\frac{\partial\eta_{1p}^*}{\partial x^*} = A(f+j)e^{fx^*+jx^*} + 2Bfe^{2fx^*} + 2Cje^{2jx^*} \quad (37)$$
$$\frac{\partial^2\eta_{1p}^*}{\partial x^{*2}} = A(f+j)^2e^{fx^*+jx^*} + 4Bf^2e^{2fx^*} + 4Cj^2e^{2jx^*}$$

Type equation here.

So that:

$$3A(f+j)^2e^{fx^*+jx^*} + 12Bf^2e^{2fx^*} + 12Cj^2e^{2jx^*} + AS_f(f+j)e^{fx^*+jx^*} + 2BfS_fe^{2fx^*} + 2CjS_fe^{2jx^*} + Aa^2e^{fx^*+jx^*} + Ba^2e^{2fx^*} + Ca^2e^{2jx^*} =$$
$$-6f^2C_1^{\ 2}e^{2fx^*} - \left(3f^2C_1C_2 + 6fC_1jC_2 + 3j^2C_2C_1\right)e^{fx^*+jx^*} - 6j^2C_2^{\ 2}e^{2jx^*} \quad (38)$$

collect the same elements on the left side

$$\left(3A(f+j)^2 + AS_f(f+j) + Aa^2\right)e^{fx^*+jx^*} + \left(12Bf^2 + 2BfS_f + Ba^2\right)e^{2fx^*} + \left(12Cj^2 + 2CjS_f + Ca^2\right)e^{2jx^*} \quad (39)$$

by balancing the coefficients in equation (38) we get

$$A = -\frac{3f^2C_1C_2 + 6fC_1jC_2 + 3j^2C_2C_1}{3(f+j)^2 + S_f(f+j) + a^2}$$
$$B = -\frac{6f^2C_1^{\ 2}}{12f^2 + 2fS_f + a^2} \quad (4.40)$$
$$C = -\frac{6j^2C_2^{\ 2}}{12j^2 + 2jS_f + a^2}$$

So, the particular solution is

$$\eta_1^*(x^*) = \frac{3f^2C_1C_2 + 6fC_1jC_2 + 3j^2C_2C_1}{3(f+j)^2 + S_f(f+j) + a^2}e^{fx^*+jx^*} - \frac{6f^2C_1^{\ 2}}{12f^2 + 2fS_f + a^2}e^{2fx^*} -$$

$$\frac{6j^2{C_2}^2}{12j^2+2jS_f+a^2}e^{2jx^*} \quad (41)$$

The general solution of equation (34) is a homogeneous solution plus a nonhomogeneous solution so that we get:

$$\eta_1^* = \eta_{1h}^* + \eta_{1p}^*$$

$$\eta_1^*(x^*) = C_3e^{fx^*} + C_4e^{jx^*} + Ae^{fx^*+jx^*} + Be^{2fx^*} + Ce^{2jx^*} \quad (42)$$

provided that the first-order boundary conditions $O(\varepsilon^1)$ are $\eta_1^*(0)=0$ and $\eta_1^*(1000)=0$. Substituting these boundary conditions into the general solution of equation (42), we get

$$\eta_1^*(0) = C_3e^0 + C_4e^0 + Ae^0 + Be^0 + Ce^0 = 0 \text{ dan}$$
$$\eta_1^*(1000) = C_3e^{1000f} + C_4e^{1000j} + Ae^{1000(f+j)} + Be^{2000f} + Ce^{2000j} = 0$$

So that,

$$C_3 = -C_4 - A - B - C \text{ dan}$$
$$C_4 = \frac{(A+B+C)e^{1000f} - Ae^{1000(f+j)} - Be^{2000f} - Ce^{2000j}}{e^{1000j} - e^{1000f}} \quad (43)$$

$$C_3 = \frac{-(A+B+C)e^{1000f} - Ae^{1000(f+j)} - Be^{2000f} - Ce^{2000j}}{e^{1000j} - e^{1000f}} - A - B - C \quad (44)$$

To obtain a solution of the second-order equation, substituting equations (42) and (32) and their derivatives to the right side of equation (20), we get:

$$\begin{aligned}
&(3C_1C_3f^2 + 3f^2C_3C_1 + 6f^2C_3C_1)e^{2fx^*} + (3C_2C_3f^2 + 3C_1C_4j^2 + 3j^2C_3C_2)e^{fx+jx^*} + \\
&+ (3C_4C_1f^2 + 6jfC_2C_3 + 6fjC_1C_4)e^{fx+jx^*} + (3C_2C_4j^2 + 3C_4C_2j^2 + 6j^2C_2C_4)e^{2jx^*} + \\
&(3C_1(f+j)^2A + 12C_2Bf^2 + 3Af^2C_1 + 3Bj^2C_2 + 6fC_1(f+j)A + 12jBC_2f)e^{2fx^*+jx^*} \\
&+ (3C_2(f+j)^2A + 12C_1Cj^2 + 3Aj^2C_2 + 3CC_1f^2 + 6jC_2(f+j)A + 12fC_1Cj)e^{fx^*+2jx^*} + \\
&(12C_1Bf^2 + 3Bf^2C_1 + 12f^2C_1B)e^{3fx^*} + (12C_1Bf^2 + 3Bf^2C_1 + 12f^2C_1B)e^{3fx^*}
\end{aligned} \quad (45)$$

The homogeneous solution of equation (20) is

$$\eta_{2h}^* = C_5e^{fx} + C_6e^{jx}$$

From the simplification of the nonhomogeneous form of equation (20), it can be assumed that the nonhomogeneous solution is of the form:

$$\eta_{2p}^* = He^{2fx^*} + Ke^{fx^*+jx^*} + Le^{2jx^*} + Me^{2fx^*+jx^*} + Ne^{fx^*+2jx^*} + Ze^{3fx^*} + Qe^{3jx^*} \quad (46)$$

the derivative of equation (46) is:

$$\frac{\partial \eta^*_{2p}}{\partial x^*} = 2fHe^{2fx^*} + (f+j)Ke^{fx^*+jx^*} + 2jLe^{2jx^*} + (2f+j)Me^{2fx^*+jx^*}$$
$$+ (f+2j)Ne^{fx^*+2jx^*} + 3fZe^{3fx^*} + 3jQe^{3jx^*} \quad (47)$$
$$\frac{\partial^2 \eta^*_{2p}}{\partial x^{*2}} = 4f^2He^{2fx^*} + (f+j)^2Ke^{fx^*+jx^*} + 4j^2Le^{2jx^*} +$$
$$(2f+j)^2Me^{2fx^*+jx^*} + (f+2j)^2Ne^{fx^*+2jx^*} + 9f^2Ze^{3fx^*} + 9j^2Qe^{3jx^*}$$

Substitute equations (46) and (47) into equation (20) so that the equation on the left side is:

$$\left(12f^2H + 2fHS_f + a^2H\right)e^{2fx^*} + \left(3(f+j)^2K + (f+j)KS_f + a^2K\right)e^{fx^*+jx^*} +$$
$$\left(12j^2L + 2jLS_f + a^2L\right)e^{2jx^*} + \left(3(2f+j)^2M + (2f+j)MS_f + a^2M\right)e^{2fx^*+jx^*} +$$
$$\left(3(f+2j)^2N + (f+2j)NS_f + a^2N\right)e^{fx^*+2jx^*} + \left(27f^2Z + a^2Z + 3fZS_f\right)e^{3fx^*} +$$
$$\left(27j^2Q + 3jQS_f + a^2Q\right)e^{3jx^*}$$

The simplification of the equation on the right side is:

$$-12C_1C_3f^2e^{2fx^*} - 3C_2C_3f^2e^{fx+jx^*} - 3C_1C_4j^2e^{fx+jx^*} - 3j^2C_3C_2e^{fx+jx^*} - 3C_4C_1f^2e^{fx+jx^*} -$$
$$6jfC_2C_3e^{fx+jx^*} - 6fjC_1C_4e^{fx+jx^*} - 12C_2C_4j^2e^{2jx^*} - 3C_1(f+j)^2Ae^{2fx^*+jx^*} - 12C_2Bf^2e^{2fx^*+jx^*}$$
$$-3Af^2C_1e^{2fx^*+jx^*} - 3Bj^2C_2e^{2fx^*+jx^*} - 6fC_1(f+j)Ae^{2fx^*+jx^*} - 12jBC_2fe^{2fx^*+jx^*} -$$
$$3C_2(f+j)^2Ae^{fx^*+2jx^*} - 12C_1Cj^2e^{fx^*+2jx^*} - 3Aj^2C_2e^{fx^*+2jx^*} - 3CC_1f^2e^{fx^*+2jx^*} -$$
$$6jC_2(f+j)Ae^{fx^*+2jx^*} - 12fC_1Cje^{fx^*+2jx^*} - 27Bf^2C_1e^{3fx^*} - 27j^2C_2Ce^{3jx^*}$$

By balancing the function coefficients on the right and left sides, we get

$$H = -\frac{12C_1C_3f^2}{12f^2 + 2fS_f + a^2}$$
$$K = -\frac{3C_2C_3f^2 + 3C_1C_4j^2 + 3j^2C_3C_2 + 3C_4C_1f^2 + 6jfC_2C_3 + 6fjC_1C_4}{3(f+j)^2 + (f+j)S_f + a^2}$$
$$L = -\frac{12C_2C_4j^2}{12j^2 + 2jS_f + a^2}$$
$$M = -\frac{3C_1(f+j)^2A + 12C_2Bf^2 + 3Af^2C_1 + 3Bj^2C_2 + 6fC_1(f+j)A + 12jBC_2f}{3(2f+j)^2 + (2f+j)S_f + a^2}$$
$$N = -\frac{3C_2(f+j)^2A + 12C_1Cj^2 + 3Aj^2C_2 + 3CC_1f^2 + 6jC_2(f+j)A + 12fC_1Cj}{3(f+2j)^2 + (f+2j)S_f + a^2}$$
$$Z = -\frac{27C_1Bf^2}{27f^2 + a^2 + 3fS_f}$$
$$Q = -\frac{27C_2Cj^2}{27j^2 + 3jS_f + a^2} \quad (48)$$

The general solution to an equation of the second order is as follows:

$$\eta^*_2(x^*) = \eta^*_{1h} + \eta^*_{1p}$$

Thus, a specific solution is obtained

$$\eta_2^*(x^*) = C_5e^{fx^*} + C_6e^{jx^*} + He^{2fx^*} + Ke^{fx^*+jx^*} + Le^{2jx^*} + Me^{2fx^*+jx^*} + Ne^{fx^*+2jx^*} + Ze^{3fx^*} + Qe^{3jx^*} \quad (49)$$

provided that the first-order boundary conditions $O(\varepsilon^2)$ are $\eta_2^*(0) = 0$ and $\eta_2^*(1000) = 0$. Substituting these boundary conditions into a homogeneous solution of equation (49) we get

$\eta_2^*(0) = C_5 + C_6 + H + K + L + M + N + Z + Q = 0$ dan

$$\eta_2^*(1000) = C_5e^{1000f} + C_6e^{1000j} + He^{2000f} + Ke^{1000(f+j)} + Le^{2000j} + Me^{1000(2f+j)} + Ne^{1000(f+2j)} + Ze^{3000f} + Qe^{3000j} = 0$$

Therefore :

$$C_5 = -C_6 - (H + K + L + M + N + Z + Q)$$

and

$$C_6 = \frac{(H+K+L+M+N+Z+Q)e^{1000f} - He^{2000f} - Ke^{1000(f+j)} - Le^{2000j} - Me^{1000(2f+j)} - Ne^{1000(f+2j)} - Ze^{3000f} - Qe^{3000j}}{e^{1000j} - e^{1000f}}$$

So,

$$C_5 = \frac{-(H+K+L+M+N+Z+Q)e^{1000f} + He^{2000f} + Ke^{1000(f+j)} + Le^{2000j} + Me^{1000(2f+j)} + Ne^{1000(f+2j)} + Ze^{3000f} + Qe^{3000j}}{e^{1000j} - e^{1000f}} - (H+K+L+M+N+Z+Q)$$

From the above calculations, a second-order solution is obtained, namely:

$$\eta_2^* = C_5e^{fx} + C_6e^{jx} + He^{2fx^*} + Ke^{fx^*+jx^*} + Le^{2jx^*} + Me^{2fx^*+jx^*} + Ne^{fx^*+2jx^*} + Ze^{3fx^*} + Qe^{3jx^*} \quad (50)$$

The solution is only limited to the second-order so that the resulting dimensionless approximation solution is as follows:

$$\eta^*(x^*) = \varepsilon^0\left(C_1e^{fx^*} + C_2e^{jx^*}\right) + \varepsilon^1\left(C_3e^{fx} + C_4e^{jx} + Ae^{fx^*+jx^*} + Be^{2fx^*} + Ce^{2jx^*}\right) + \varepsilon^2\left(C_5e^{fx} + C_6e^{jx} + He^{2fx^*} + Ke^{fx^*+jx^*} + Le^{2jx^*} + Me^{2fx^*+jx^*} + Ne^{fx^*+2jx^*} + Ze^{3fx^*} + Qe^{3jx^*}\right) \quad (51)$$

Equation (51) is then converted into a dimensional form based on Table 2 so that it is obtained:

$$\eta(x) = h_0\,\eta^*(x^*) \quad (52)$$

Then substitute equation (52) into equation (21):

$$\eta(x,t) = h_0\,\eta^*(x^*)e^{i\omega t} \quad (53)$$

Based on the data in the Estuary of the KarangMumus, the Mahakam River, the parameters are set according to Table 2.

Table 2. Parameter values in theestuary of the KarangMumus,Mahakam river

Parameters	**Value**	**Unit**
h_0	10	m
T	21600	s
π	3.14	-
Omega ($2\pi/T$)	0.000291	$1/s$

Highest tide	1.55	m
ManningsCoefficient (n) / sediment	0.03	$s/m^{1/3}$
x	10000	m
a	0.000297	-
g	9.8	m/s^2
$S_f = (n^2/h^{\frac{1}{3}})\, g$ (h highest in the estuary)	0.003902	-
ε	0.000297	-
Lowest tide	-1.55	m
h at the mouth of the estuary	11.55	m

The following curve shows the propagation of tidal wave tides to the end of the estuary,

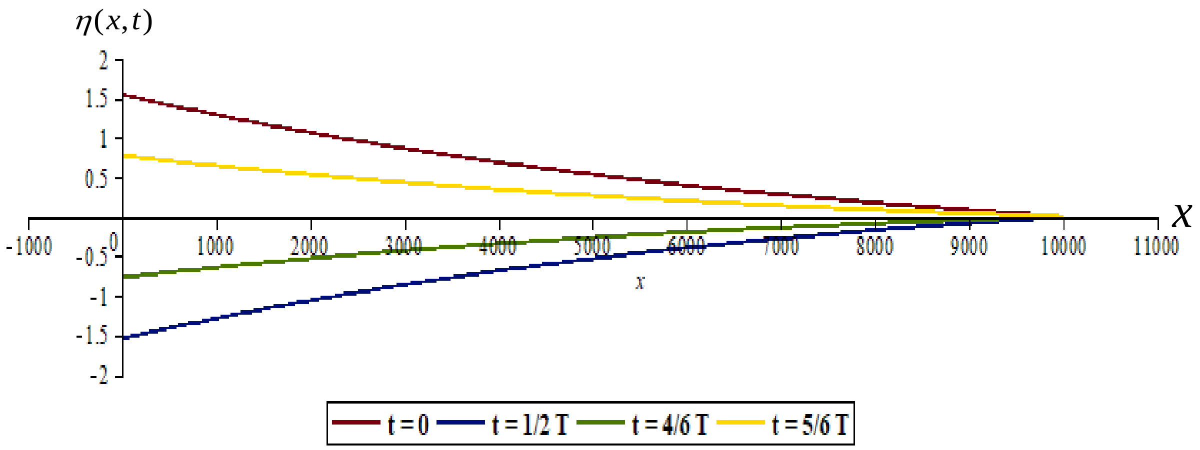

Figure2. Deviation curve $\eta(x,t)$ against distance x in different t with $x = 10000$meters, and $T = 21600$seconds

Figure 2 shows the movement of the tides in different seas. On the slope of the riverbed with a distance of meters measured from the mouth of the estuary to the end of the estuary, the tidal water comes at a deviation of 1.55 meters then decreases and the wave amplitude is also getting smaller, so that at a distance of 10000 meters. After 10000 meters it is estimated that there will be no more tides. This is likely to be strongly influenced by the basic friction between tidal currents and the bottom topography. The effect of basic friction will also change the shape of the tidal wave [11]. The basic friction will reduce the tidal wave energy [12]. The energy loss due to basic friction then causes the tidal wave height to continue to decrease. This event usually occurs in shallow waters (estuaries) where the velocity is constant. The decrease in height and the loss of tidal wave energy due to bottom friction is referred to as tidal wave damping or dumping. Strengthening due to resonance can occur when the tidal period approaches or corresponds to the natural period of the river mouth.

6. Conclusion

The analytical solution of non-linear PDP, especially the Saint Venant in the one-dimensional equation, has been solved using perturbation methods. A cross-differentiation between conservation of mass and conservation of momentum equations results in the hyperbolic differential equation. By taking a solution from zero-order in the perturbation method, then the hyperbolic PDP can be determined by using the variable separation method. The solution is also reviewed through the completion of graph visualization. The solution of zero-order gives decreasing amplitude of the tidal wave to the end of the estuary as we expected beforehand. The deviation of water level from mean water level gives the potential of brackish water to be benefited to develop aquaculture activities.

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